Towards primitive data types for Coq: 63-bits integers and persistent arrays[1]

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Motivation

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But also efficiency of extracted code: floating-point arithmetic,…

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- Bytecode-based dedicated compiler and virtual machine
- Compilation to native code through OCaml

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In the following, I will review the solutions implemented in Coq for efficient integer arithmetic and suggest some improvements.

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```
Coq < Definition x := 10000.
Warning: Stack overflow or segmentation fault happens when
working with large numbers in nat (observed threshold may
vary from 5000 to 70000 depending on your system limits
and on the command executed).
x is defined
```

Numbers in Coq: binary numbers

```
Inductive positive : Set :=
  | xI : positive -> positive
  | xO : positive -> positive
  | xH : positive.

Inductive N : Set :=
  | N0 : N
  | Npos : positive -> N.
```

Slight complication for uniqueness of `0`, definition of positive binary numbers (`positive`) and then binary naturals (`N`).

Exponential gain in space and time, but still too limited for heavy computations.

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Leaves were first implemented by an inductive type with 256 constructors, but in recent versions they are substituted with native 31-bits integers.

Numbers in Coq: 31-bits integers

In the current version of Coq, 31-bits integers are represented by an inductive type:

```
Inductive digits : Type := D0 | D1.

Definition digits31 t :=
  Eval compute in nfun digits 31 t.

Inductive int31 : Type := I31 : digits31 int31.
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This machinery is called “retroknowledge”.

Retroknowledge vs primitive data types

An alternative to "retroknowledge" would be to introduce in the formalism a primitive type `int`, with operators and axiomatized equational theory.

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An alternative to “retroknowledge” would be to introduce in the formalism a primitive type `int`, with operators and axiomatized equational theory.

With a major gain: benefits from machine arithmetic are not limited to the conversion test, but available in the whole system.

But a few drawbacks:

- Requires to enrich the formalism
- Does not give computational meaning to the axioms encoding the equational theory

Retroknowledge vs primitive data types

Still, we propose to replace “retroknowledge” by some primitive data types, because it is sometimes unavoidable:

- No guarantee that an inductive type can be defined to reflect the computational behavior
- Explicit constructions can be too costly to typecheck (outside conversion), or even to allocate!

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We ported them as a primitive data type:

```
Register array : Type -> Type as array_type.
Register get : forall {A:Type}, array A -> int -> A
   as array_get.
Register set : forall {A:Type}, array A -> int -> A
   -> array A as array_set.
Register length : forall {A:Type}, array A -> int as
    array_length.
Register init : forall {A:Type}, int -> (int -> A)
   -> A -> array A as array_init.
Register map : forall {A B:Type}, (A -> B) -> array
   A -> array B as array_map.
```

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We propose to do it the other way around, and provide two implementations of an interface of unsigned 63-bits arithmetic:

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The implementation is chosen when Coq is compiled.

Primitive 63-bits integers

We plugged the standard kernel conversion and the reduction by compilation to native code to this native 63-bits arithmetic (the VM will follow).

As a byproduct, we started to re-think the design of the `BigN` library.

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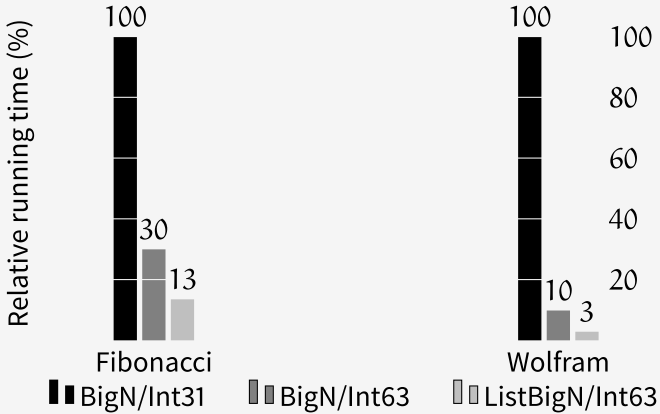
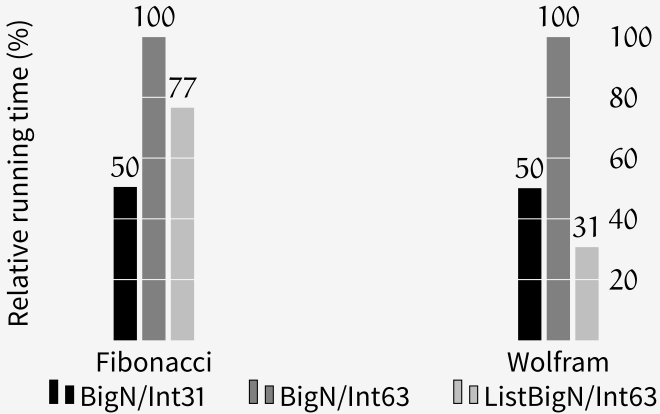
We implemented a prototype library based on lists of 63-bits integers:

```
Definition big_nat := list int.
Fixpoint add_big (a b : big_nat) (c : bool) :=
  match a, b with
   | nil, nil => if c then 1 :: nil else nil
   | i :: is, nil => if c then succ a else a
   | nil, j :: js => if c then succ b else b
   | i :: is, j :: js => if c then
     let r := i + j + 1 in r :: add_big is js (r <= i)
     else let r := i + j in r :: add_big is js (r < i)
  end.
Definition add (a b : big_nat) := add_big a b false.
```

Benchmarks (64-bits architecture)



Benchmarks (32-bits architecture)



Conclusion

Three proposals:

- Replace “retroknowledge” with primitive data types
- Switch from 31-bits to 63-bits arithmetic
- Design a simpler and more efficient library for big numbers

Moral of the story:

- Better verify a computation with a slightly bigger trusted base than not verify it at all
- Often valuable to periodically re-think the design of parts of a system (here Coq) to follow the evolution of hardware and software components

Conclusion

Remaining issues:

- Modular and flexible extension of the trusted base
- Unverified ad-hoc code in the parser and the printer for big numbers

Thank you!