A New Formalization of Power Series in Coq *

Catherine Lelay

Inria Saclay Île-de-France, Toccata, Palaiseau, F-91120
LRI, Université Paris-Sud, CNRS, Orsay, F-91405

To define elementary functions, proof assistants usually formalize power series. In Coq, exponential function, sine and cosine are defined using these, but only theorems that are strictly needed to define them are proved. Most proof assistants with a library of real analysis (C-CoRN, Isabelle/HOL, HOL Light, HOL4, Mizar, etc) provide power series, but they are hardly comprehensive with respect to differentiability and integrability.

Using limits formalized in a previous work [1], I have made a user-friendly definition of power series. I have checked its usability by proving relations on Bessel functions defined as

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{p=0}^{+\infty} \frac{(-1)^p}{p!(n+p)!} \left(\frac{x}{2}\right)^{2p} \quad (1)$$

These functions were also studied in HOL Light [2], but in an other form, optimized to compute approximations of its zeros.

I have chosen to prove results on power series using more general theorems. For example, results about differentiability of power series are proved using theorems of differentiability about sequences of functions.

Section 1 presents elements of [1] useful to formalize power series and Section 2 presents my new approach in Coq and various results about series of real numbers and sequences of functions.

1 Limits

As a power series is the limit of a partial sums $\left(\sum_{k=0}^{n} a_n x^n\right)_{n\in\mathbb{N}}$, we need either a specific notion of convergence for series (as it is done in the standard library), or a notion of limits for sequence (as in C-CoRN and HOL based systems). Contrarily to the approach used in the Coq standard library, my definition of limit of sequences [1] is practical because it is a total function that does not require a proof term from the user:

```
Definition Lim_seq (u : nat → R) : R := (limsup (u_n) + liminf (u_n)) / 2
```

$$\texttt{Definition Lim\_seq (u : nat} \to \texttt{R) : R :=} \ \frac{\overline{\lim}\,(u_n) + \underline{\lim}\,(u_n)}{2}$$

Moreover, main results about convergence were already proved for sequences and, thanks to properties on $\overline{\lim}$ and $\underline{\lim}$, scalar multiplication and several other rewritings do not need hypotheses of convergence.

I have extended this library with properties about sub-sequences and proved rewritings without hypotheses for the usual sub-sequences:

```
Lemma Lim_seq_incr_n (u : nat → R) (N : nat) : Lim_seq (fun n ⇒ u (n + N)) = Lim_seq u.
```

Some convergence criteria have also been added to ease proofs of convergence for series.

Using this formalization of limits, I have defined power series as a total function in a natural way:

```
Definition PSeries (a : nat → R) (x : R) := Lim_seq (sum_f_R0 (fun k ⇒ a k * x ^ k)).
```

2 Power Series

2.1 Convergence and divergence

An important notion about power series needed for theorems of analysis is the radius of convergence:

$$\mathcal{C}_a = \sup\left\{ r \in \mathbb{R} \;\middle|\; \sum |a_n r^n| \text{ is convergent} \right\}.$$

*This research was supported by the Coquelicot project of the Digiteo cluster and the Île-de-France regional council.

As this definition is not suitable for all needed theorems, I have also proved that the radius of convergence $\mathcal{C}_a$ is equal to $\sup \{r \in \mathbb{R} \mid \exists M \in \mathbb{R},\ \forall n \in \mathbb{N}, |a_n r^n| \leq M\}$. Using extended real numbers $\mathbb{R} \cup \{\pm\infty\}$ defined in [1], most theorems can be proved without separating the cases where $\mathcal{C}_a$ is finite or not.

I have also proved d'Alembert's Criterion for series and have used it to prove a version for power series:

$$\lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \ell \quad \Rightarrow \quad \left(\forall x \in \mathbb{R}, \left(\ell = 0 \vee |x| < \ell^{-1}\right) \quad \Rightarrow \quad \sum_{n \in \mathbb{N}} a_n x^n \text{ is convergent} \right).$$

Moreover, I have proved that $\mathcal{C}_a = \ell^{-1}$ if $\ell \neq 0$ and $\mathcal{C}_a = +\infty$ otherwise. I have used d'Alembert's criterion to check that Formula (1) is well-defined for all real numbers.

2.2 Arithmetic operations

I have first defined addition and scalar multiplication for series and proved corresponding theorems for power series. I have used them to verify the equality

$$\forall x \in \mathbb{R},\ J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x).$$

The multiplication of power series by a variable has also been proved using scalar multiplication for series, but using additional theorems about index shift for sequences.

2.3 Differentiability

Differentiability of power series may be proved in the restricted scope of power series, but these results are also consequences of more general theorems about sequences of functions. Notions of uniform and normal convergence for sequences of functions exist in Coq, but the hypotheses they require are too complex. Moreover, only theorems about continuity are proved using this formalism. So, I have chosen to redefine uniform convergence and prove the needed theorems.

Using theorems about differentiability, I have proved that my definition of Bessel functions is a solution of the Bessel equation:

$$\forall x \in \mathbb{R},\ x^2 \cdot J_n''(x) + x \cdot J_n'(x) + \left(x^2 - n^2\right) J_n(x) = 0.$$

Each case (J_0, J_1, and J_{n+2}) needs only 20 lines of proof.

3 Conclusion

This work amounts to 30 definitions, 112 lemmas, and about 3000 lines of Coq. No axioms other than those from the standard library are needed for this development. The dependence on excluded middle leaks from theorems of standard library and will soon be corrected to only use axioms postulating the existence of real numbers. It is available at `http://coquelicot.saclay.inria.fr/`. It will soon be extended with other operations as multiplication, composition, and integration of power series. The next steps in my construction of a user-friendly library of real analysis for Coq will be the formalization of equivalents and generalized limits.

References

[1] Sylvie Boldo, Catherine Lelay, and Guillaume Melquiond. Improving real analysis in Coq: a user-friendly approach to integrals and derivatives. In Chris Hawblitzel and Dale Miller, editors, *Proceedings of the Second International Conference on Certified Programs and Proofs*, volume 7679 of *Lecture Notes in Computer Science*, pages 289–304, Kyoto, Japan, December 2012.

[2] John Harrison. Fast and accurate Bessel function computation. In Javier D. Bruguera, Marius Cornea, Debjit DasSarma, and John Harrison, editors, *Proceedings of the 19th IEEE Sympoisum on Computer Arithmetic*, pages 104–113, Portland OR, 2009. IEEE Computer Society.